THE STEADY FLOW OF A WEAKLY IONIZED GAS BETWEEN PARALLEL PLATES ASSUMING ANISOTROPIC CONDUCTIVITY

(USTANOVIVSHEESIA TECHENIE SLABO IONIZOVANNOGO GAZA MEZHDU PARALLELNYMI PLASTINAMI S UCHETOM ANISOTROPII PROVODIMOSTI)

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It is well known that if the condition $\omega\tau \ll 1$ is not fulfilled in an ionized gas (where ω is the cyclotron frequency of a charged particle and τ is the average time between collisions (strong magnetic field, rarefied medium)), it is essential to take account of the fact that the transfer coefficients depend on the magnetic field, and they become anisotropic [1, 5-9]. In the last few years several publications have appeared dealing with the flow of ionized gas, assuming anisotropic coefficients of viscosity and conductivity. Articles [2,3] give derivations of the fundamental magnetic plasmo-dynamic equations taking into account the influence of the magnetic field on the transfer process within a fully ionized gas, and a detailed study of Couette flow is adduced. Kaplan [4] and Lighthill [11] have studied the effect of anisotropic conductivity on a magneto-gasdynamic shock wave and on the wave motion of a conducting medium. Article [10] dealt with the effect of anisotropic conductivity on the longitudinal flow of a weakly ionized gas in a narrow annular channel within a radial magnetic field.

This paper deals with steady flow of a weakly ionized gas between parallel non-conducting plates within a transverse homogeneous magnetic field B_0. The following assumptions are made in order to simplify the problem:

1) the mean free path in the gas, λ, is much less than the transverse dimension of the channel $2a$, i.e. the conditions of a homogeneous medium are fulfilled;

2) the degree of ionization of the gas is small;

3) the condition $\omega_i\tau_i \ll 1$ is fulfilled for the ions, and this allows

the viscosity coefficient η to be considered a scalar quantity, and the effect of ion slip with respect to the gas can be neglected;

4) the following inequality is satisfied: $R_m = \sigma \mu V_c a \ll 1$, where V_c is the mean gas velocity in the channel; conditions (2), (3) and (4) allow one to use a stream function in the following form:

$$\mathbf{j} + \omega\tau (\mathbf{j} \times \mathbf{k}) = \sigma (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \qquad \left(\omega\tau = \frac{B_0 e\tau}{m_e}\right)$$

(where k is the unit vector in the direction of the magnetic field)

5) the physical characteristics of the gas (ρ, η, σ, μ, ϵ) are assumed constant.

With these assumptions the fundamental equations for this problem take the form

$$\rho (\mathbf{v} \nabla) \mathbf{v} = - \nabla p + \mathbf{j} \times \mathbf{B} + \eta \Delta \mathbf{v}, \qquad \operatorname{div} \mathbf{v} = 0 \tag{1}$$

$$\operatorname{rot} \mathbf{H} = \mathbf{j}, \qquad \operatorname{div} \mathbf{H} = 0, \qquad \mathbf{j} + \omega\tau (\mathbf{j} \times \mathbf{k}) = \sigma (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \tag{2}$$

$$\operatorname{rot} \mathbf{E} = 0, \qquad \operatorname{div} \varepsilon \mathbf{E} = \vartheta \tag{3}$$

We will assume that the homogeneous external magnetic field $B_0 = \mu H_0$ is parallel to the z-axis, whilst there are constant pressure gradients along the x- and y-axes $(\partial p/\partial x) = - P_x$, $(\partial p/\partial y) = - P_y$ and there is an external homogeneous electric field E_{0x} and E_{0y}. All equations of the system (1) to (3) can then be satisfied if we assume that the required quantities are functions of the transverse coordinate z only. The system therefore reduces to the following equations:

$$\frac{d^2 u}{dz^2} + \frac{B_0}{\eta} j_y + \frac{P_x}{\eta} = 0, \qquad \frac{d^2 v}{dz^2} - \frac{B_0}{\eta} j_x + \frac{P_y}{\eta} = 0, \qquad \frac{\partial p}{\partial z} = j_x B_y - j_y B_x \tag{4}$$

$$j_x = - \frac{dH_y}{dz}, \qquad j_y = \frac{dH_x}{dz} \tag{5}$$

$$j_x + \omega\tau j_y = \sigma (E_{0x} + B_0 v), \qquad j_y - \omega\tau j_x = \sigma (E_{0y} - B_0 u) \tag{6}$$

$$E_z = B_x v - B_y u, \qquad dE_z / dz = \vartheta / \varepsilon \tag{7}$$

$$E_x = E_{0x}, \qquad E_y = E_{0y}, \qquad B_z = B_0, \qquad j_z = 0, \qquad w = 0 \tag{8}$$

We find from (6)

$$\begin{aligned} j_x &= \frac{\sigma}{1 + \omega^2\tau^2} [(E_{0x} - \omega\tau E_{0y}) + B_0 (\omega\tau u + v)] \\ j_y &= \frac{\sigma}{1 + \omega^2\tau^2} [(E_{0y} + \omega\tau E_{0x}) + B_0 (\omega\tau v - u)] \end{aligned} \tag{9}$$

Using (9) to eliminate the stream density from (4), we arrive at the following system of equations for the velocity components:

$$\frac{d^2u}{dz^2} - \frac{N^2}{a^2} u + \omega\tau \frac{N^2}{a^2} v = - \frac{P_x}{\eta} - \frac{N^2}{B_0 a^2}(E_{0y} + \omega\tau E_{0x})$$
$$\frac{d^2v}{dz^2} - \frac{N^2}{a^2} v - \omega\tau \frac{N^2}{a^2} u = - \frac{P_y}{\eta} + \frac{N^2}{B_0 a^2}(E_{0x} - \omega\tau E_{0y}) \tag{10}$$

$$M^2 = \frac{B_0^2 a^2 \sigma}{\eta}, \qquad N^2 = \frac{M^2}{1+\omega^2\tau^2} \tag{11}$$

The solution of (10), which is even with respect to the z-coordinate, is of the form

$$u = C_1 \cosh\frac{\alpha N z}{a} \cos\frac{\beta N z}{a} + C_2 \sinh\frac{\alpha N z}{a} \sin\frac{\beta N z}{a} + A_1 \tag{12}$$

$$v = - C_2 \cosh\frac{\alpha N z}{a} \cos\frac{\beta N z}{a} + C_1 \sinh\frac{\alpha N z}{a} \sin\frac{\beta N z}{a} + A_2 \tag{13}$$

where

$$\alpha = \frac{1}{\sqrt{2}}\sqrt{\sqrt{1+\omega^2\tau^2}+1}, \qquad \beta = \frac{1}{\sqrt{2}}\sqrt{\sqrt{1+\omega^2\tau^2}-1} \tag{14}$$

$$A_1 = \frac{1}{B_0^2\sigma}(P_x + \omega\tau P_y) + \frac{E_{0y}}{B_0}, \qquad A_2 = \frac{1}{B_0^2\sigma}(P_y - \omega\tau P_x) - \frac{E_{0x}}{B_0} \tag{15}$$

If we determine the constants C_1 and C_2 using the boundary conditions

$$u = v = 0 \qquad \text{for } z = a \tag{16}$$

we find

$$C_1 = - \frac{A_1 \cosh\alpha N \cos\beta N + A_2 \sinh\alpha N \sin\beta N}{\Delta_1}, \qquad \Delta_1 = \sinh^2\alpha N + \cos^2\beta N$$
$$C_2 = - \frac{A_1 \sinh\alpha N \sin\beta N - A_2 \cosh\alpha N \cos\beta N}{\Delta_1}. \tag{17}$$

For volume gas fluxes Q_x and Q_y in the directions of the x- and y-axes, we obtain by calculation

$$Q_x = \int_{-a}^{a} u\,dz = 2a\left(A_1 - \frac{A_1\Phi_1 + A_2\Phi_2}{M\Delta_1}\right)$$
$$Q_y = \int_{-a}^{a} v\,dz = 2a\left(A_2 - \frac{-A_1\Phi_2 + A_2\Phi_1}{M\Delta_1}\right) \tag{18}$$

where

$$\Phi_1 = \alpha \sinh\alpha N \cosh\alpha N + \beta \sin\beta N \cos\beta N$$
$$\Phi_2 = \beta \sinh\alpha N \cosh\alpha N - \alpha \sin\beta N \cos\beta N \tag{19}$$

By use of Formula (9) we derive the following expressions for the full flows in the directions x and y:

$$
\begin{aligned}
I_x &= \int_{-a}^{a} j_x\,dz = \frac{2a\sigma}{1+\omega^2\tau^2}\left[(E_{0x}-\omega\tau E_{0y}) + \frac{B_0}{2a}(\omega\tau Q_x + Q_y)\right] \\
I_y &= \int_{-a}^{a} j_y\,dz = \frac{2a\sigma}{1+\omega^2\tau^2}\left[(E_{0y}+\omega\tau E_{0x}) + \frac{B_0}{2a}(-Q_x + \omega\tau Q_y)\right]
\end{aligned}
\tag{20}
$$

For known velocity components $u(z)$ and $v(z)$, by using (5) and (9), the induced magnetic-field intensities H_x and H_y can be found. The induced electric-field strength E_z and the volumetric charge density ϑ are determined from Equation (7).

We now deal with two particular cases.

1) Assume that a given constant channel pressure gradient P_x along the x-axis maintains the steady motion of the gas, whilst $Q_y = I_y = E_{0x} = 0$. This example corresponds to the problem of the flow of a conducting gas through an infinitely long channel with non-conducting walls whose width is large compared to the height. The conditions $Q_y = I_y = 0$ yield the equation

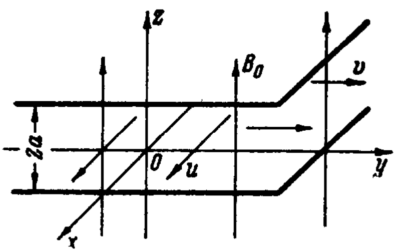

Fig. 1.

$$
A_2 - \frac{-A_1\Phi_2 + A_2\Phi_1}{M\Delta_1} = 0, \qquad E_{0y} - B_0\frac{Q_x}{2a} = 0 \tag{21}
$$

It follows from these latter expressions that $I_x = 0$.

From system (21), using (15), we find

$$
A_1 = \frac{M\Delta_1(M\Delta_1 - \Phi_1)}{N^2\Delta_2}\frac{P_x a^2}{\eta}, \qquad A_2 = \frac{M\Delta_1\Phi_2}{N^2\Delta_2}\frac{P_x a^2}{\eta} \tag{22}
$$

where

$$
\Delta_2 = \Phi_1(M\Delta_1 - \Phi_1) + \Phi_2(M\Delta_1\omega\tau - \Phi_2) \tag{23}
$$

The flow in the x-axis direction is given by the formula

$$
Q_x = \frac{(M\Delta_1 - \Phi_1)^2 + \Phi_2^2}{N^2\Delta_2}\frac{2P_x a^3}{\eta} \tag{24}
$$

which for the case M = const, $\omega\tau \to 0$ leads to the well-known solution of the Hartmann problem with isotropic conductivity and total current equal to zero:

$$Q_x = \frac{2P_x a^3}{\eta} \frac{M \cosh M - \sinh M}{M^2 \sinh M} \tag{25}$$

The graphs in Fig. 2 illustrate the relation between the flow $Q_x^\circ = Q_x/(2P_x a^3 : \eta)$ and the quantity $\omega\tau$ given by Formula (24) for various values of the Hartmann number M. If only the intensity of the external magnetic field H_0 is varied, the quantity $\gamma = \omega\tau/M$ remains

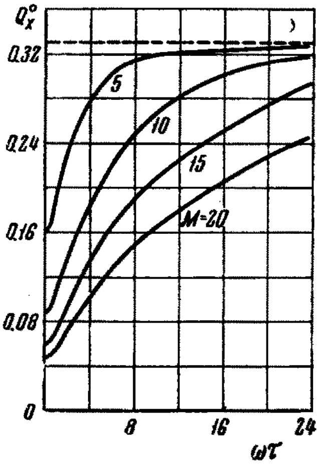

Fig. 2.

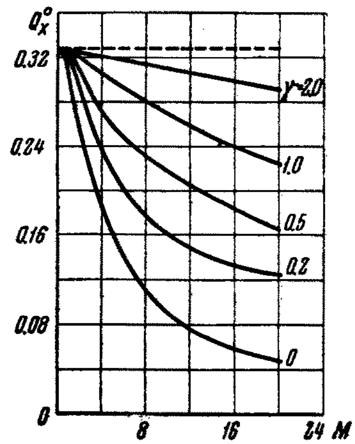

Fig. 3.

constant and is a characteristic of the degree of rarefaction of the medium. A graph showing the relation between the flow Q_x° and the Hartmann number is shown in Fig. 3. Despite the fact that the total flow Q_y and total current I_y are zero, because of the anisotropic character of the conductivity transverse flows and electric currents arise.

2) In the second example it is assumed that both the medium and the currents can flow freely in the x- and y-directions. For simplicity, put $P_y = E_{0y} = E_{0x} = 0$, whilst the value of P_x is assumed to be given. We then find

$$A_1 = \frac{1}{M^2} \frac{P_x a^2}{\eta}, \qquad A_2 = -\frac{\omega\tau}{M^2} \frac{P_x a^2}{\eta} = -\omega\tau A_1 \tag{26}$$

$$Q_x = \frac{2P_x a^3}{\eta} \frac{(M\Delta_1 - \Phi_1) + \omega\tau\Phi_2}{M^3\Delta_1}, \qquad I_x = \frac{2P_x a}{B_0} \frac{\Phi_2}{M\Delta_1}$$

$$Q_y = -\frac{2P_x a^3}{\eta} \frac{\omega\tau(M\Delta_1 - \Phi_1) - \Phi_2}{M^3\Delta_1}, \qquad I_y = -\frac{2P_x a}{B_0} \frac{M\Delta_1 - \Phi_1}{M\Delta_1} \tag{27}$$

In the limiting case M = const, $\omega\tau \to 0$, the expressions found transform into formulas of another regime of motion in the Hartmann problem, in which the intensity of the electric field is equal to zero:

$$Q_x = \frac{2P_x a^3}{\eta} \frac{M\cosh M - \sinh M}{M^3 \cosh M}, \qquad I_y = -\frac{2P_x a}{B_0} \frac{M\cosh M - \sinh M}{M\cosh M} \tag{28}$$

A graph showing the ratio between the flow Q_y arising because of anisotropic conductivity of the transverse stream and the main flow Q_x as a function of $\omega\tau$ is shown in Fig. 4. Lateral flow vanishes both when $\omega\tau \to 0$ and for large values of $\omega\tau$.

It follows from the illustrations on the graphs and from analysis of the formulas that at large values of $\omega\tau$ the solutions obtained tend asymptotically to the normal ones which correspond to the problem for a non-conducting fluid:

$$u = \frac{P_x a^2}{\eta}\frac{1}{2}\left[1 - \left(\frac{z}{a}\right)^2\right], \quad Q_x = \frac{2P_x a^3}{\eta}\frac{1}{3} \tag{29}$$

Thus the effect of a magnetic field on the main stream is weakened as the value of $\omega\tau$ increases because of the decrease in conductivity of the gas. On the other hand, anisotropic conductivity complicates the whole flow pattern, and this is expressed by the appearance of transverse flows, an induced electric field, and a volumetric electric charge.

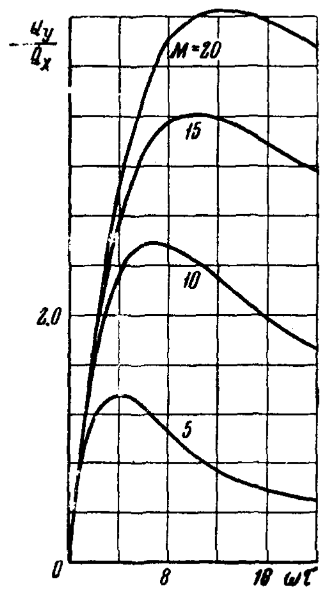

Fig. 4.

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